A Theory of Space Based on the Notion of Convexity

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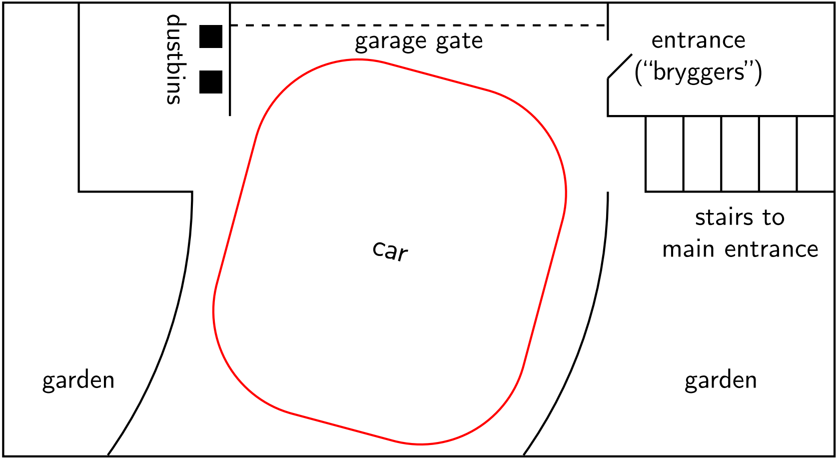
11. maj 2015

Introduction: Geometry and IT

Imagine . . .

- . . . that we had, instead of our old Volvo, an autonomous car.
- Moving into our garage, then, would not be so easy a task for that car.

Moving our car into the garage



The task

The autonomous car would have to avoid collisions with

1. the low walls to the left and right of the path leading up to the garage and marking off that path from the garden;
2. the stairs leading up to the official entrance at the main floor;
3. the door at the entrance to the ground floor;
4. the left boundary wall of the house, which also bounds the garage to the lef.

The taks is further complicated by the fact that the path into the garage is curved rather than straight and that both the bath and the garage are rather narrow for a modern car.

Geometry

- It is rather obvious, that the autonomous car must have a pretty large amount of geometric knowledge in order to manage the task.
- In physics, movements of rigid bodies are often modelled by movements of "mass points", . . .
- . . . and the paths of such mass points are calculated by means of analytical geometry (and the laws of physics, of course).
- It is clear that this method is not very helpful for the present problem.

Some Background Geometry

Background 1: Kinds of Geometry

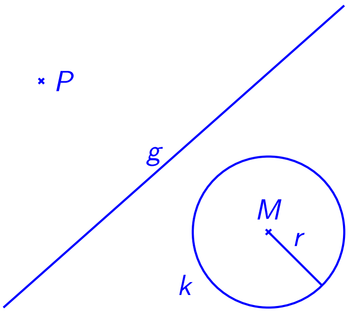
Two kinds of geometry

The first systematic presentation of geometry is Euclid's (ca. 300 BC) *Elements*. In this book, Euclid presents the subject as synthetic geometry. This is an axiomatic theory dealing with points, lines, planes, etc. It is concerned with the construction of figures whose properties are demonstrated by logical deductions.

Analytic geometry analyses geometric entities by algebraic means. It goes back to Fermat (160?–1665) and Descartes (1596–1650). "Algebra" in their times meant, of course, numerical algebra.

Synthetic and analytic geometry

Synthetic Geometry



Analytic Geometry

(x_p, y_P)

$y = ax + c$

pow(r,2) = pow((x-xM),2) + pow((y-yM),2)

Leibniz' complaints

- Analytical geometry of the Descartes-style is a rather indirect ("per circuitum") detour to geometry.
- It does not directly confront such geometrically important notions as, for instance, that of shape ("forma") and similarity ("similitudo").
- Leibniz asks therefore for a kind of algebraic geometry which directly "calculates" with points, lines, etc. as arithmetic and algebra (at his times) does with numbers.

Algebra *vs.* geometry

Group theory

G1 $a \circ (b \circ c) = (a \circ b) \circ c$

G2 $a \circ e = a = e \circ a$

G3 $a \circ a^{-1} = e = a^{-1} \circ a$

Special laws

Id $a \circ a = a$

Co $a \circ b = b \circ a$

Affine geometry

A1 $P \neq Q \rightarrow \overset{1}{\exists} g.P, Q \mathrel{\mathrm{I}} g$

A2 $\overset{1}{\exists} g.[P \mathrel{\mathrm{I}} g \wedge g \parallel h]$

A3 $\exists P, Q, R.[P \neq Q \wedge P \neq R \wedge Q \neq R \wedge \neg \exists g.P, Q, R \mathrel{\mathrm{I}} g]$

Requirements

We want to have a geometry which

1. directly deals with important geometric concepts,
2. but nevertheless is as algebraic in character as possible.

Background 2: Geometrically important concepts

What do we require of a geometrically important concept?

1. It should refer to real entities — i. e.,entities which really can be found in space — rather than to fictions and/or abstractions.
2. It should be fruitful — i. e., it should be useful in defining further concepts and in stating many facts.
3. It should be cognitively relevant — i. e., it should play some role for our conception of space.
4. It should be useful — i. e., it should have applications in various contexts.

A problematic case: points

- The notion of a point is explained in Euclid's first definition of the first book of his *Elements*: *"A point is that which has no part."*
- Despite its seeming simplicity, the legitimization of the notion of a point has always been questioned:
  1. *"Plato even used to object to this class of things as a geometrical fiction"*; Aristotle *Metaphysics* 992^{a} 20.
  2. The stoics assumed *". . . that such bounding elements of solids only have a wafer-thin existence in thinking"*; Proclus Lycaeus *A Commentary on the First Book of Euclid's Elements*, p. 228.
  3. *"The space of geometry and physics consists of an infinite number of points, but no one has ever seen or touched a point"*; Bertrand Russell *Our Knowledge of the External World*, p. 119.

Psychological evidence

According to Edgar Rubin, Russell is not right:

1. *"Man har ikke bemærket, at om end breddeløse Streger ikke kan tænkes som Naturgenstande, kan de dog sanseligt anskueligt opleves"*; *Synsoplevede figurer*, p. 180.
2. *"Ligesom der findes breddeløse Linjer, findes der udstrækningsløse Punkter"*; loc. cit., p. 181

More psychology

Otto Selz argued in 1930, that the point is — as "der reine Ort im Raum" —

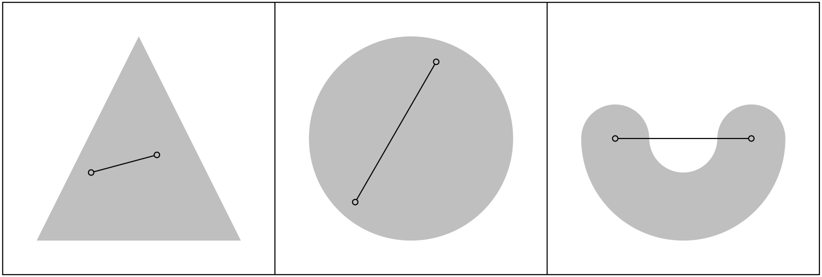
3. *"strukturgesetzlich [. . .] gefordert [. . .] und es ist eine relativ untergeordnete Frage, ob das empirische Minimum Visibile, die Punktgestalt, als ein reines Ortsphänomen anzusehen oder als ein winziges rundes flächenartiges Gebilde zu betrachten ist"*; *Die Struktur der Steigerungsreihen und die Theorie von Raum, Zeit, und Gestalt*, p. 40.

Conclusion 1

Therefore I shall not worry about using the concept of a point in my system of geometry.

Convexity

Definition: A region of space is convex iff it contains with each two points also the straight line segment which connects these them.



Convexity is an important concept

- It plays a central role in various subdisciplines of mathematics (such as functional analysis, game theory, linear programming, etc.).
- It has important applications in computational geometry (collision avoidance, shape analysis, data analysis).
- Quite generally, many geometric concepts can be defined in terms of convexity and many geometric facts can be stated with the help of that concept.
- It seems also to play a major role in the human cognitive system.

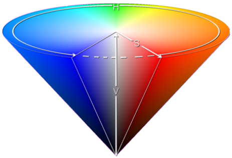
Convexity in the human cognitive system

Gärdenfor's thesis:

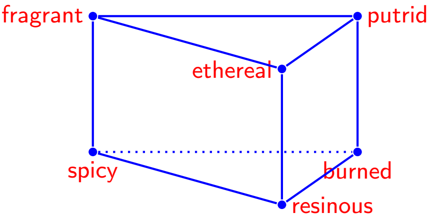
$$\boxed{\text{A (genuine) property is a convex region of a quality space.}}$$

Example of quality spaces

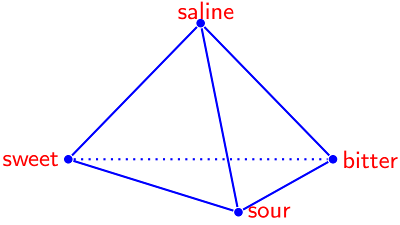
Runge's Colour Sphere (1810)



Colour Cone



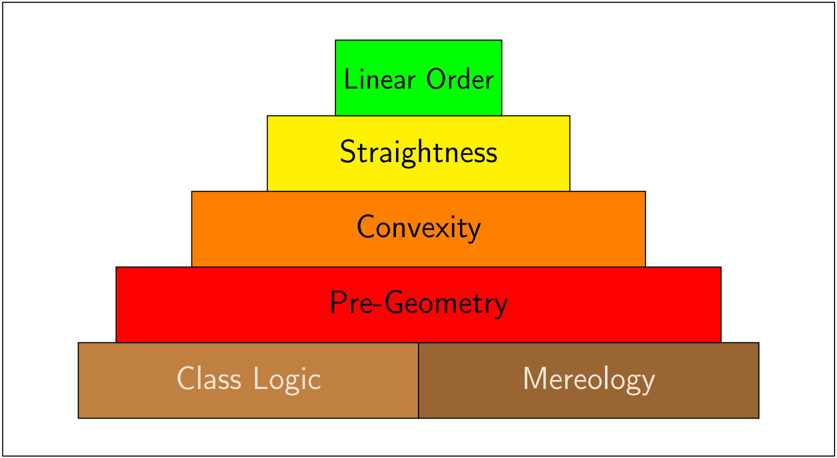
Henning's Smell Prims (1915)



Henning's Taste Tetrahedron (1916)

Presenting the System

Geometry stack



The Mereological Background

Reconsidering Euclid's definition of a point

- Σημεῖόν ἐστιν, οὗ μέρος οὐθέν. — "A point is that which has no part."
- Here the concept of a point is defined in terms of the part-of-relationship.
- The theory of this relationship is called mereology (gr. μέρος 'part').
- Modern mereology has its origin in the work of the Polish logician Stanisław Leśniewski.

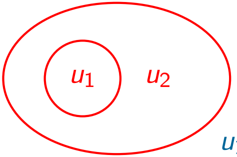
Defining points in mereology

- In modern mereology, one conceives of the part-of-relationship as a reflexive relation: Everything is (trivially) a part of itself.
- In symbols: $u \mathrm{P} u$.
- Points are thus only parts of themselves:

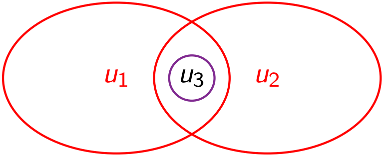
$$p \in \mathrm{D}^{\mathbf{p}} \underset{\text{def}}{\Longleftrightarrow} u \mathrm{P} p \rightarrow u = p$$

- Or the original Euclid: $p \in \mathrm{D}^{\mathbf{p}} \underset{\text{def}}{\Longleftrightarrow} \neg \exists u.[u \neq p \wedge u \mathrm{P} p]$.

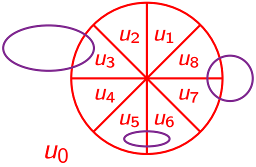
Mereological concepts



$u_1 \mathrm{P} u_2$



$u_1 \mathrm{O} u_2 \underset{\text{def}}{\Longleftrightarrow} \mathrm{P}^{>} u_1 \cap \mathrm{P}^{>} u_2 \neq \varnothing$



- $\{u_1, u_2, \dots, u_8\} \subseteq \mathrm{P}^{>} u_0$
- $v \mathrm{O} u_0 \to v \mathrm{O} u_m$ for one m with $1 \leqslant m \leqslant 8$
- $\Sigma(u_0, \{u_1, u_2, \dots, u_8\})$, i. e., u_0 is the sum of $\{u_1, u_2, \dots, u_8\}$

Defining the mereological sum

- u is the sum (or: supremum) of A iff
  1. each element of A is a part of u
  2. each individual overlapping with u overlaps with some element of A.
- $\Sigma(u, A) \underset{\text{def}}{\Longleftrightarrow} A \subseteq \mathrm{P}^{>}u \wedge \forall u_1 \in \mathrm{P}^{>}u . \exists u_2 \in A . u_1 \mathrm{O} u_2$
- $\sup(A) \underset{\text{def}}{=\!=} \iota u . \Sigma(u, A)$
- $u_1 + u_2 \underset{\text{def}}{=\!=} \sup(\{u_1, u_2\})$

Tarski's axioms of atomistic mereology

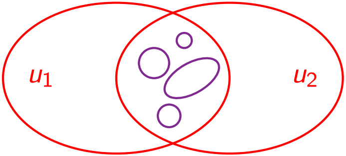
MER 2 $u_1 \mathrm{P} u_2 \wedge u_2 \mathrm{P} u_3 \rightarrow u_1 \mathrm{P} u_3$
The part-of-relation is transitive.

MER 3 $\Sigma(u_1, \{u_2\}) \rightarrow u_1 = u_2$
The sum of a singleton equals its sole element.

MER 4 $A \neq \varnothing \rightarrow \exists u.\Sigma(u, A)$
Every non-empty class has a sum.

MER 6 $\mathrm{P}^{>} u \cap \mathrm{D}^{\mathbf{p}} \neq \varnothing$
Each individual has a punctual part.

Intersection and multiplication



$$\Pi(u_0, \{u_1, u_2\}) \Leftrightarrow \Sigma(u_0, \mathrm{P}^{>} u_1 \cap \mathrm{P}^{>} u_2)$$

The general definition:

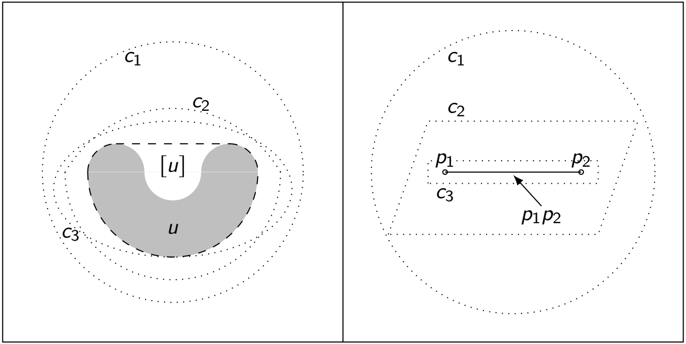
- $\Pi(u, A) \underset{\text{def}}{\iff} \Sigma(u, \{u_1 \mid \forall u_2 \in A. u_1 \mathrm{P} u_2\})$
- $\inf(A) \underset{\text{def}}{=\!=} \imath u.\Pi(u, A)$
- $u_1 \cdot u_2 \underset{\text{def}}{=\!=} \inf(\{u_1, u_2\})$

First Steps into Geometry

"The hull"

- Let $\mathbb{D}^{\mathbf{c}}$ be the domain of convex regions.
- Each region u is a part of some element of $c \in \mathbb{D}^{\mathbf{c}}$.
- The smallest convex region thus containing u is called h's convex hull and is denoted by "$[u]$".
- The convex hull of u is thus a convex approximation towards u.
- You should think about the convex hull of a region as a tight "rubber wrap" wrapping this region.

The idea

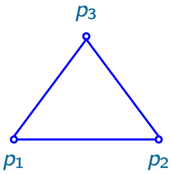


The definitions encapsulating the idea

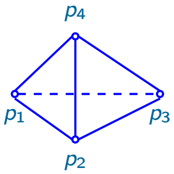
- $[u] \overset{}{\underset{\text{def}}{=\!=}} \inf(c \mid u \in \mathrm{P}^{>} c)$ — the convex hull of u.
- $[p_1, p_2, \ldots, p_n] \underset{\text{def}}{=\!=} [\sup(p_1, p_2, \ldots, p_n)]$ — the *n-tope* spanned by $p_1, p_2, \ldots, p_n$.

Examples of n-topes

a 2-tope,
i. e., a line segment
$p_1 p_2 = [p_1, p_2]$



a 3-tope,
i. e., a triangle
$\triangle p_1 p_2 p_3 = [p_1, p_2, p_3]$



a 4-tope,
i. e., a tetrahedron
$\mathrm{Tetra}\, p_1 p_2 p_3 p_4 = [p_1, p_2, p_3, p_4]$

Segments and convex regions

- We can now fix the domain D^{s} of (straight) line segments: $\mathrm{D}^{\mathsf{s}} = \{u \mid \exists p_1 p_2 . u = p_1 p_2\}$.
- We can prove,
  1. that the class $\mathrm{Cv} \xlongequal[\mathrm{def}]{} \{u \mid \forall p_1, p_2 \in \mathrm{P}^{>} u . p_1 p_2 \mathrm{P} u\}$ actually has the properties which we expect to hold true for the domain D^{c} of convex regions;
  2. and even $\mathrm{Cv} \subseteq \mathrm{D}^{\mathsf{c}}$.
- Of course, we would like to have the converse inclusion, too. Hence we adopt the axiom:

$$\text{GEO 1} \quad \forall p_1, p_2 \in \mathrm{P}^{>} u . p_1 p_2 \mathrm{P} u \rightarrow u \in \mathrm{D}^{\mathsf{c}}$$

Segments and points

- We did <u>not</u> require that the boundary points of a segment are distinct from each other.
- However, it seems <u>natural</u> to assume that a segment pp shrinks down to p — the <u>idempotent</u> law.
- Idempotency implies that points are convex. Conversely, the convexity of points implies the idempotent law: $p = pp$.
- Hence we assume the axiom: Points are convex.

GEO 2 $\mathrm{D}^{\mathsf{p}} \subseteq \mathrm{D}^{\mathsf{c}}$

Aside on algebra

- We just noted the idempotent law for the join operation. Let us generalize that relation thus:
  $u_1 u_2 \stackrel{\text{def}}{=\!=} \mathsf{sup}(s \mid \exists u_3 \in \mathrm{P}^{>} u_1, u_4 \in \mathrm{P}^{>} u_2 . s = u_3 u_4)$.
- It is then very easy to see that for this operation
  1. idempotent law holds true: $u_1 u_1 = u_1$
  2. and that the commutative law does, too: $u_1 u_2 = u_2 u_1$.
- In the full system of geometry the
  3. assosiative law $u_1(u_2 u_3) = (u_1 u_2) u_3$

  is also valid for this operation.
- We thus approach an algebraic form of geometry.

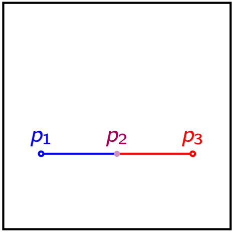
Straightness

Segments should be straight

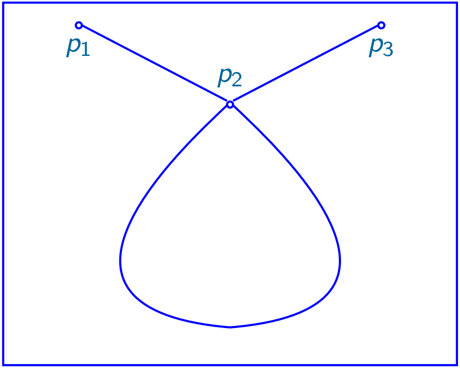
- We expect the line segments fixed by converging classes of convex regions to have certain properties.
- Especially, they should be "straight" rather than "curved".
- Therefore we add two axioms which explicate what "straightness" means and which require the segments to have this property.

Decomposability

(GEO 5) An internal point of a segment dissects that segment into two complementary segments with the division point as the sole common part.
$p_2 \mathrm{P} p_1 p_3 \rightarrow p_1 p_3 = p_1 p_2 + p_2 p_3 \wedge p_2 = p_1 p_2 \cdot p_2 p_3$



Excluded: Loops



Line segments do not include loops.

Non-Bending

GEO 6 The sum of two segments sharing two points is again a segment.
$\exists p_1, p_2.[p_1 \neq p_2 \wedge p_1, p_2 \in \mathrm{P}^{>} s_1 \cap \mathrm{P}^{>} s_2] \to s_1 + s_2 \in \mathrm{D}^{\mathsf{s}}$